BEST APPROXIMATION FOR THE k-DIGAMMA FUNCTIONS

Abstract. In this paper, we show a best approximation for the k-digamma functions.

1. Introduction

The Euler gamma function is defined all positive real numbers x by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

It is common knowledge that the logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function, and $\psi^{(m)}(x)$ for $m \in \mathbb{N}$ are known as the polygamma functions. The gamma, digamma and polygamma functions play an important role in the theory of special functions, and have many applications in other many branches, such as statistics, fractional differential equations, mathematical physics and theory of infinite series. some of the work about the complete monotonicity, convexity and concavity, and inequalities of these special functions may refer to [1, 2, 3, 4, 6, 7, 8, 9, 15, 16, 17, 18].

In 2007, Diaz and Pariguan [5] defined the $k-$analogue of the gamma function for $k > 0$ and $x > 0$ as

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{x(x+k)\cdots(x+(n-1)k)},$$

where $\lim_{k\to 1} \Gamma_k(x) = \Gamma(x)$. Similarly, we may define the $k-$analogue of the digamma and polygamma functions as

$$\psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x) \quad \text{and} \quad \psi_k^{(m)}(x) = \frac{d^m}{dx^m} \psi_k(x).$$

It is well known that the k-analogues of the polygamma functions satisfy the following integral and series identities (see [12])

$$\begin{aligned}\psi_k^{(m)}(x) &= (-1)^{m+1} m! \textstyle\sum_{n=0}^\infty \frac{1}{(nk+x)^{m+1}} \\ &= (-1)^{m+1} \textstyle\int_0^\infty \frac{1}{1-e^{-kt}} t^m e^{-xt} dt.\end{aligned} \tag{1.1}$$

For more properties of these functions, the reader may see the references [11, 12, 13].

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies $(-1)^n f^n(x) \geq 0$ for $x \in I$ and $n \geq 0$. A characterization of completely monotonic functions is given by the Bernstein-Widder theorem which reads that a function $f(x)$ on $x \in [0;\infty)$ is completely monotonic if

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and only if there exists a bounded and non-decreasing function $g(t)$ such that the integral

$$f(x)=\int_0^{\infty}e^{-xt}dg(t)$$

converges for $x\in[0;\infty)$. That is, a function $f(x)$ is completely monotonic on $x\in[0;\infty)$ if and only if it is a Laplace transform of a bounded and non-decreasing measure $g(t)$. From above theorem it follows that completely monotonic functions on $[0;\infty)$ are always strictly completely monotonic unless they are constant (see [14]).

In [10], Mortici gave better approximation of the form

$$\psi\left(x\right)\sim\ln\left(x+a\right)-\frac{1}{bx}.$$

Motivated by this work, we natural study best approximation of the k-digamma. The objective of this note is to find the appropriate constant a and b, and such that the approximation formula

$$\psi_k\left(x\right)\sim\frac{1}{k}\ln\left(\frac{x}{k}+a\right)-\frac{1}{bx}-\frac{\ln k}{k}$$

is best.

2. Main results

Lemma 2.1. ([19, formula (12)]) Let $r>0$. Then

$$\frac{1}{x^r}=\frac{1}{\Gamma(r)}\int_0^{\infty}t^{r-1}e^{-xt}dt. \tag{2.1}$$

Theorem 2.1. *(1) Let* $B_k\left(x\right)=\psi_k\left(x\right)-\frac{1}{k}\ln\left(\frac{x}{k}+\frac{1}{\sqrt{6}}\right)-\frac{1}{\left(6-2\sqrt{6}\right)x}-\frac{\ln k}{k}$. *If* $0<k\leq 1$, *then the function* $-B_k(x)$ *is completely monotonic on* $(0,\infty)$.
(2) If $C_k\left(x\right)=\psi_k\left(x\right)-\frac{1}{k}\ln\left(\frac{x}{k}-\frac{1}{\sqrt{6}}\right)+\frac{1}{\left(6+2\sqrt{6}\right)x}-\frac{\ln k}{k}$, *then the function* $C_k(x)$ *is completely monotonic on* $\left(\frac{k}{\sqrt{6}},\infty\right)$ *and* $k\in(0,1]$.

Proof. Define

$$F_{a,b,k}=\psi_k\left(x\right)-\frac{1}{k}\ln\left(\frac{x}{k}+a\right)+\frac{1}{bx}-\frac{\ln k}{k}.$$

Using (1.1) and Lemma 2.1, we get

$$\psi_k'\left(x\right)=\int_0^{\infty}\frac{te^{-xt}}{1-e^{-kt}},$$

$$\frac{1}{x}=\int_0^{\infty}e^{-xt}dt,$$

and

$$\frac{1}{x}=\int_0^{\infty}e^{-xt}dt.$$

Furthermore, we easily obtain

$$\begin{aligned}F'_{a,b,k}\left(x\right)&=\int_0^{\infty}\frac{te^{-xt}}{1-e^{-kt}}dt-\int_0^{\infty}ke^{-(x+ka)t}dt-\frac{1}{b}\int_0^{\infty}te^{-xt}dt\\&=\int_0^{\infty}\frac{e^{-(x+ka)t}}{e^{kt}-1}\phi_{a,b,k}\left(t\right)dt\end{aligned}$$

where

$$\phi_{a,b,k}(t) = te^{k(a+1)t} - ke^{kt} + k - \frac{t\left(e^{k(a+1)t} - e^{kat}\right)}{b}.$$

By developing in power series, we have

$$\begin{aligned}\phi_{a,b,k}(t) &= \left(1-k^2\right)t + k\left(a+1-\tfrac{1}{b}-\tfrac{k^2}{2}\right)t^2 \\ &+k^2\left(\tfrac{(a+1)^2}{2}-\tfrac{k^2}{6}-\tfrac{2a+1}{2b}\right)t^3 \\ &+k^{n-1}\sum_{n=4}^{\infty}\tfrac{(b-1)n(a+1)^{n-1}+na^{n-1}-bk^2}{b\cdot n!}t^n.\end{aligned}$$

Since $k \in (0,1]$, we get

$$\begin{aligned}\phi_{a,b,k}(t) &\geq \left(1-k^2\right)t + k\left(a+\tfrac{1}{2}-\tfrac{1}{b}\right)t^2 + k^2\left(\tfrac{(a+1)^2}{2}-\tfrac{1}{6}-\tfrac{2a+1}{2b}\right)t^3 \\ &+\cdots+k^{n-1}\sum_{n=4}^{\infty}\tfrac{(b-1)n(a+1)^{n-1}+na^{n-1}-b}{b\cdot n!}t^n.\end{aligned}$$

We put

$$\begin{cases} a+\frac{1}{2}-\frac{1}{b}=0, \\ \frac{(a+1)^2}{2}-\frac{1}{6}-\frac{2a+1}{2b}=0, \end{cases}$$

with the solution $a = \pm\frac{1}{\sqrt{6}}, b = 6 \mp 2\sqrt{6}$. For $a = \frac{1}{\sqrt{6}}, b = 6-2\sqrt{6}$, we have

$$\phi_{a,b,k}(t) \geq \left(1-k^2\right)t + k^{n-1}\sum_{n=4}^{\infty}\frac{(b-1)\,n(a+1)^{n-1}+na^{n-1}-b}{b\cdot n!}t^n.$$

Since $\left(5-2\sqrt{6}\right)n\left(1+\frac{1}{\sqrt{6}}\right)^{n-1} + n\left(\frac{1}{\sqrt{6}}\right)^{n-1} - \left(6-2\sqrt{6}\right) > 0$ for $n \geq 4$ and $1-k^2 > 0$, we get $\phi_{a,b,k}(t) > 0$. This implies that the function $\phi'_{a,b,k}(t)$ is completely monotonic on $(0,\infty)$.

Considering to

$$\frac{1}{k}\ln x - \frac{1}{x} < \psi_k(x) < \frac{1}{k}\ln x$$

and $\lim_{x\to\infty} B_k(x) = 0$, we get

$$B_k(x) < B_k(\infty) = 0.$$

So, the proof of part (1) is complete. Completely similar, we also prove the part (2).

□

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