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Study of α^*-Homeomorphisms by α^*-closed sets

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Type of Article

Abstract

In this paper, we introduce a new kind of closed sets called α^*-closed sets in a topological space and investigate their properties. These closed sets are compared with the closed sets and the generalized closed sets. We also introduce the α^*-homeomorphisms and develop their properties by using the α^*-closed maps and α^*-continuous maps.

Keywords: g-closed set, g-continuous function, g-irresolute map, g-homeomorphism.

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1 Introduction

The concept of generalized closed sets called $g-$closed sets were introduced by Levine [1] in 1970 and investigated their properties. With the introduction of this generalized closed sets, many authors introduced different type of generalized closed sets and studied their properties. The ω-closed set [2], semi-generalized closed (briefly sg-closed) set [3], generalized α-closed (briefly $g\alpha$-closed) set [4], regular generalized closed (briefly rg-closed) set [5], beta weakly generalzed closed (briefly βwg-closed) set [6], generalized semi open-closed (briefly gso-closed) set [7] are some of the generalized closed sets in the literature.
Homeomorphisms are mappings which preserves the topological properties of the given topological spaces. By definition, a homeomorphism between topological spaces X and Y is a bijective map $f : X \to Y$ when both f and f^{-1} are continuous. For the generalization of the notion of homeomorphisms, Maki etal [8] introduced and studied the g-homeomorphisms and gc-homeomorphisms between topological spaces. Devi etal [9] introduced and studied sg-homeomorphisms and gs-homeomorphisms. Veera kumar [10] introduced and studied $*g$-homeomorphisms and $*gc$-homeomorphisms.There are some recent researches carried out on generalized homeomorphisms [11,12,13,14,15].

In this paper, we first introduced a new kind of generalized closed sets called the α^*-closed sets and studied their topological properties. The α^*-closed sets are compared with the closed sets and the $g-$ closed sets. We also introduced the α^*-closed maps and α^*-continuous maps and investigated their properties. The notion of irresoluteness was introduced by Crossely and Hilderband [16] in 1972 which is independent of continuous maps. In this paper, we introduced

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the α^*-irresolute and investigated this with the α^*-continuous maps. Finally, we define the notion of α^*-homeomorphism and studied the properties of α^*-homeomorphism in a general topological space.

2 Preliminaries

Throughout this paper, we represent X, Y and Z as the topological spaces (X, τ), (Y, σ) and (Z, η) respectively on which no separation axioms are assumed unless otherwise stated. For a subset A of X, $cl(A)$ denotes the closure of A and $int(A)$ denotes the interior of A.

We recall the following definitions in the topological space X.

Definition 2.1. (1) A subset A of a space X is said to be generalized closed (g-closed) set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2.2. (8) A map $f : X \to Y$ is said to be g-closed map if for each closed set F in X, $f(F)$ is g-closed in Y.

Definition 2.3. (17) A map $f : X \to Y$ is said to be generalized continuous (g-continuous) map if $f^{-1}(V)$ is g-open in X for each open set V in Y.

Definition 2.4. (18) A bijective function $f : X \to Y$ is called generalized homeomorphism (g-homeomorphism) if both f and f^{-1} are g-continuous.

3 α^*-Closed Set

Definition 3.1. A subset A of a space X is said to be a α^*-closed set if int(cl(A)) $\subseteq U$ whenever $A \subseteq U$ and U is open in X.

From the definition, it is clear that every closed set is a α^*-closed set as well as every g-closed set is a α^*-closed set.

Example 3.1. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$ be a topology on X. Then, $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}$ and X are the α^*-closed sets. Moreover, $\phi, \{a\}, \{a, b\}, \{a, c\}$ and X are the g-closed sets.*

Example 3.2. *In $\mathbb{R}^n$ space with usual topology, every closed interval is a α^*-closed set.*

Theorem 3.3. *The intersection of two α^*-closed sets in a space X is a α^*-closed set in X.*

Proof. Let A and B be two α^*-closed sets. Then, $int(cl(A)) \subseteq U_1$ and $int(cl(B)) \subseteq U_2$ whenever $A \subseteq U_1$ and $B \subseteq U_2$ for the open sets U_1 and U_2 in X. Now, $int(cl(A)) \cap int(cl(B)) \subseteq U_1 \cap U2$ whenever $(A \cap B) \subseteq U_1 \cap U2$ and $U_1 \cap U2$ is open in X. Since $int(cl(A \cap B)) \subseteq int(cl(A)) \cap int(cl(B))$, $A \cap B$ is a α^*-closed set in X. □

The union of two α^*-closed sets in a space X need not be a α^*-closed set in X. This can be seen from the following example.

Example 3.4. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$. Then, $A = \{a\}$ and $B = \{c\}$ are α^*-closed in X; but, $A \cup B = \{a, c\}$ is not a α^*-closed set in X.*

In general, the collection of all α^*-closed sets in X does not form a topology for X because the arbitrary union of α^*-closed sets is not a α^*-closed set in X as seen in the above example.

Definition 3.2. A topological space X is a T_{α^*}-space if every α^*-closed set in X is a closed set in X.

Theorem 3.5. *In T_{α^*}-space, the finite union of α^*-closed sets is a α^*-closed set.*

Proof. Suppose $A = \cup_i^n A_i$ is a finite union of α^*-closed sets in T_{α^*}-space. Then,

$$A^c = (\cup_i^n A_i)^c = \cap_i^n A_i^c.$$

Since in T_{α^*} space, every α^*-closed set is a closed set, so A_i^c open for each i and so A^c is open. Therefore, A is closed and hence α^*-closed. □

4 α^*-closed map

Definition 4.1. A map $f : X \to Y$ is said to be α^*-closed map if for each closed set F in X, $f(F)$ is a α^*-closed set in Y.

Definition 4.2. A map $f : X \to Y$ is said to be α^*-open map if for each open set U in X, $f(U)$ is a α^*-open set in Y.

Definition 4.3. A map $f : X \to Y$ is said to be α^*-continuous map if $f^{-1}(V)$ is α^*-closed in X for each closed set V in Y.

Definition 4.4. A map $f : X \to Y$ is said to be a α^*-irresolute if $f^{-1}(V)$ is a α^*-closed in X for each α^*-closed set V in Y.

Lemma 4.1. *Every closed map is a α^*-closed map.*

Proof. Let $f : X \to Y$ be a closed map and let F be a closed set in X. Then, $f(F)$ is a closed set in Y and so α^*-closed Y. Thus, f is a α^*-closed map. □

The converse of the above Lemma need not be true in general.

Example 4.2. *Let $X = Y = \{a, b, c\}$ and let $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, c\}\}$ be topologies on X and Y respectively. Let $f(x) = x$ for every x in X. Then, f is a α^*-closed map. As the image of $\{c\}$ is not a closed set, f is not a closed map.*

Remark 4.1. Every g-closed map is a α^*-closed map.

Lemma 4.3. *If $f : X \to Y$ is a α^*-closed map and if $A = f^{-1}(B)$ for some closed set B in Y, then $f_A : A \to Y$ is a α^*-closed map.*

Proof. Let F be a closed set in A. Then, there is a closed set H in X such that $F = A \cap H$. Then,$f_A(F) = f(A \cap H) = f(A) \cap f(H) = B \cap f(H)$. Now $f(H)$ is a α^*-closed set in Y as f is a α^*-closed map. Therefore, $B \cap f(H)$ is a α^*-closed set in Y and so f_A is a α^*- closed map. □

Theorem 4.4. *Let $f : X \to Y$ and $g : Y \to Z$ be α^*-closed maps. If f is a closed map, then $go f : X \to Z$ is a α^*-closed map.*

Proof. Let F be a closed set in X. Then, $f(F)$ is a closed set in Y as f is a closed map. Then, $(g \circ f)(F) = g(f(F))$ is a α^*-closed set in Z as g is a α^*-closed map. Therefore, $g \circ f$ is a α^*-closed map. □

Lemma 4.5. *If $f : X \to Y$ is a α^*-irresolute, then f is a α^*-continuous map.*

Proof. Let F be any closed set in Y. Since every closed set is a α^*-closed set, F is a α^*-closed set in Y. Since f is a α^*-irresolute, $f^{-1}(F)$ is a α^*-closed set in X. Hence, f is a α^*-continuous. □

Lemma 4.6. *If $f : X \to Y$ is a α^*-continuous map and Y is a T_{α^*}-space, then f is a α^*-irresolute.*

Proof. Let F be a α^*-closed set in Y. Since Y is a T_{α^*}-space, F is a closed set. Then, $f^{-1}(F)$ is a α^*-closed set in X. Hence f is a α^*-irresolute. □

Theorem 4.7. *If $f : X \to Y$ is a α^*-irresolute and $g : Y \to Z$ is a α^*-continuous map, then $gof : X \to Z$ is a α^*-continuous map.*

Proof. Let F be a closed set in Z. Then, $g^{-1}(F)$ is a α^*-closed set in Y as g is α^*-continuous. Now $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is a α^*-closed set in X as f is a α^*-irresolute. Therefore, $g \circ f$ is a α^*-continuous map. □

Corollary 4.8. *If $f : X \to Y$ and $g : Y \to Z$ are α^*-continuous maps and Y is a T_{α^*}-space, then $g \circ f$ is a α^*-continuous map.*

Proof. In T_{α^*}-space, each α^*-closed set is a closed set, the result is directly follows from theorem 4.2. □

Lemma 4.9. *Every continuous map is a α^*-continuous map.*

Proof. Let $f : X \to Y$ be a continuous map and G be an open set in Y. Then, $f^{-1}(G)$ is an open set in X and hence α^*-open set in X. Therefore, f is a α^*-continuous map. □

Remark 4.2. Every g-continuous map is a α^*-continuous map.

5 α^*-Homeomorphism

Definition 5.1. A bijection $f : X \to Y$ is called α^*- homeomorphism when f is both α^*-continuous and α^*-closed map.

Lemma 5.1. *Every homeomorphism is a α^*-homeomorphism.*

Proof. Let $f : X \to Y$ be a homeomorphism. Then, f is both continuous and closed. Then, clearly f is a α^*-continuous and α^*-closed. So f is a α^*-homeomorphism. □

Lemma 5.2. *Every g-homeomorphism is a α^*-homeomorphism.*

Proof. Let $f : X \to Y$ be a g-homeomorphism. Then, f is both g-continuous and g-closed. Then, clearly f is α^*-continuous and α^*-closed. So f is a α^*-homeomorphism. □

The converse of the above two lemmas need not be true as seen from the following example.

Example 5.3. *Let X with a topology $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and Y with a topology $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$ where $X = Y = \{a, b, c\}$. If $f : X \to Y$ with $f(a) = a,\ f(b) = c$ and $f(c) = b$. Then, f is a α^*-homeomorphism, but not a homeomorphism and also not a g-homeomorphism as the inverse image of $\{a, b\}$ in Y is not closed and also not g-closed in X.*

Theorem 5.4. *For any bijection $f : X \to Y$, the following statements are equivalent:*

(a) the inverse map $f^{-1} : Y \to X$ is a α^*- continuous map,

(b) f is a α^*-open map,

(c) *f is a α^*-closed map.*

Proof. Let $f^{-1} : Y \to X$ be a α^*-continuous map and G be any open set in X. Then, the inverse image of G under f^{-1}, $f(G)$, is α^*-open in Y and so f is a α^*-open map. Now, let f be a α^*-open map and let F be any closed set in X. Then, F^c is open in X so $f(F^c)$ is α^*-open in Y. But $f(F^c) = Y \setminus f(F)$ and so $f(F)$ is α^*- closed in Y. Therefore, f is a α^*- closed map. Finally, let f be a α^*-closed map and let F be any closed set in X. Then, $f(F)$ is α^*-closed in Y. But $f(F)$ is the inverse image of F under f^{-1}. Therefore, f^{-1} is α^*- continuous. □

Theorem 5.5. *Let $f : X \to Y$ be a α^*- continuous map from a space X onto a space Y. Then, the following statements are equivalent:*

(a) *f is a α^*-open map,*

(b) *f is a α^*-homeomorphism,*

(c) *f is a α^*-closed map.*

Proof. Assume that f is a α^*-open map. Then, clearly f is a α^*- homeomorphism. Now, if f is a α^*-homeomorphism, then, by definition f is a α^*-closed map. Finally, if f is a α^*-closed map, then, by Theorem 5.4, f is a α^*-open map. □

The following example shows that, in general, the composition of two α^*-homeomorphisms need not be a α^*-homeomorphism.

Example 5.6. *Let $X = Y = Z = \{a, b, c\}$ be topological spaces with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$ and $\eta = \{Z, \phi, \{a\}, \{b\}, \{a, b\}\}$ respectively. Let $f : X \to Y$ with $f(a) = a$, $f(b) = c$, $f(c) = b$ and let $g : Y \to Z$ with $g(x) = x$ for each x in Y. Then, both f and g are α^*-homeomorphisms, but their composition go$f : X \to Z$ is not a α^*- homeomorphism as $\{a, c\}$ is closed in Z, but (gof)$^{-1}$($\{a, c\}$) $= \{a, b\}$ is not α^*-closed in Z.*

Theorem 5.7. *Let X and Z be any two topological spaces and let Y be a T_{α^*}-space. If $f : X \to Y$ and $g : Y \to Z$ be α^*-homeomorphisms, then the composition $g \circ f : X \to Z$ is a α^*-homeomorphism.*

Proof. Let F be a closed set in Z. Then, $g^{-1}(F)$ is a α^*-closed set in Y as g is a α^*-continuous map. Since Y is a T_{α^*}-space, $g^{-1}(F)$ is a closed set in Y. Thus $f^{-1}(g^{-1}(F))$=$(g \circ f)^{-1}(F)$ is a α^*-closed set in X. Thus $g \circ f$ is a α^*-continuous map.

Again, let F be a closed set in X. Then, $f(F)$ is a α^*-closed set in Y as f is a α^*-closed map. Since Y is a T_{α^*}-space, $f(F)$ is a closed set in Y. Thus $g(f(F)) = (g \circ f)(F)$ is a α^*-closed set in Z. Thus $g \circ f$ is a α^*-closed map. Hence $g \circ f$ is a α^*-homeomorphism. □

6 Conclusions

In this paper, we introduced a new kind of generalized closed sets, α^*-closed sets, and investigated their properties. The α^*-closed maps, α^*-continuous maps and α^*-irresolutes were also defined and investigated their properties. Finally, the α^*-homeomorphisms were introduced and their properties were established.

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