Study of α^*-Homeomorphisms by α^*-closed sets

Abstract

In this paper, we introduce a new kind of closed sets called α^*-closed sets in a topological space and investigate their properties. These closed sets are compared with the closed sets and the generalized closed sets. We also introduce the α^*-homeomorphisms and developed their properties by using the α^*-closed maps and α^*-continuous maps.

Keywords: g-closed set, g-continuous function, g-irresolute map, g-homeomorphism.

2010 Mathematics Subject Classification: 54A05; 54C08

1 Introduction

The concept of generalized closed sets called $g-$closed sets were introduced by Levine [1] in 1970 and investigated their properties. With the introduction of this generalized closed sets, many authors introduced different type of generalized closed sets and studied their properties. Homeomorphism are mappings which preserves the topological properties of the given topological spaces. By definition, a homeomorphism between topological spaces X and Y is a bijective map $f : X \to Y$ when both f and f^{-1} are continuous. For the generalization of the notion of homeomorphisms, Maki etal [2] introduced and studied the g-homeomorphisms between topological spaces. There are some recent researches carried out on generalized homeomorphisms [3,4,5,6].

In this paper, we first introduced a new kind of generalized closed sets called the α^*-closed sets and studied their topological properties. The α^*-closed sets are compared with the closed sets and the $g-$ closed sets. We also introduced the α^*-closed maps and α^*-continuous maps and investigated their properties. The notion of irresoluteness was introduced by Crossely and Hilderband [7] in 1972 which is independent of continuous maps. In this paper, we introduced the α^*-irresolute and investigated this with the α^*-continuous maps. Finally, we define the notion of α^*-homeomorphism and studied the properties of α^*-homeomorphism.

2 Preliminaries

Throughout this paper, we represent X, Y and Z as the topological spaces (X, τ), (Y, σ) and (Z, η) respectively on which no separation axioms are assumed unless otherwise stated. For a subset A of X, $cl(A)$ denotes the closure of A and $int(A)$ denotes the interior of A.

We recall the following definitions in the topological space X.

Definition 2.1. Levine (1970)A subset A of a space X is said to be generalized closed (g-closed) set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

Definition 2.2. Maki (1991)A map $f : X \to Y$ is said to be g-closed map if for each closed set F in X, $f(F)$ is g-closed in Y.

Definition 2.3. Balachandran (1991)A map $f : X \to Y$ is said to be generalized continuous (g-continuous) map if $f^{-1}(V)$ is g-open in X for each open set V in Y.

Definition 2.4. Andrijevic (1986)A bijective function $f : X \to Y$ is called generalized homeomorphism (g-homeomorphism) if both f and f^{-1} are g-continuous.

3 α^*-Closed Set

Definition 3.1. A subset A of a space X is said to be a α^*-closed set if int(cl(A)) $\subseteq U$ whenever $A \subseteq U$ and U is open in X.

From the definition, it is clear that every closed set is a α^*-closed set as well as every g-closed set is a α^*-closed set.

Example 3.1. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$ be a topology on X. Then, $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}$ and X are the α^*-closed sets. Moreover, $\phi, \{a\}, \{a, b\}, \{a, c\}$ and X are the g-closed sets.*

Theorem 3.2. *The intersection of two α^*-closed sets in a space X is a α^*-closed set in X.*

Proof. Let A and B be two α^*-closed sets. Then, $int(cl(A)) \subseteq U_1$ and $int(cl(B)) \subseteq U_2$ whenever $A \subseteq U_1$ and $B \subseteq U_2$ for the open sets U_1 and U_2 in X. Now, $int(cl(A)) \cap int(cl(B)) \subseteq U_1 \cap U2$ whenever $(A \cap B) \subseteq U_1 \cap U2$ and $U_1 \cap U2$ is open in X. Since $int(cl(A \cap B)) \subseteq int(cl(A)) \cap int(cl(B))$, $A \cap B$ is a α^*-closed set in X. □

The union of two α^*-closed sets in a space X need not be a α^*-closed set in X. This can be seen from the following example.

Example 3.3. *Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a, b\}, \{a, c\}, X\}$. Then, $A = \{b\}$ and $B = \{c\}$ are α^*-closed in X; but, $A \cup B = \{b, c\}$ is not a α^*-closed set in X.*

In general, the collection of all α^*-closed sets in X does not form a topology for X because the arbitrary union of α^*-closed sets is not a α^*-closed set in X as seen in the above example.

Definition 3.2. A topological space X is a T_{α^*}-space if every α^*-closed set in X is a closed set in X.

Theorem 3.4. *In T_{α^*}-space, the finite union of α^*-closed sets is a α^*-closed set.*

Proof. Suppose $A = \cup_i^n A_i$ is a finite union of α^*-closed sets in T_{α^*}-space. Then,

$$A^c = (\cup_i^n A_i)^c = \cap_i^n A_i^c.$$

Since in T_{α^*} space, every α^*-closed set is a closed set, so A_i^c open for each i and so A^c is open. Therefore, A is closed and hence α^*-closed. □

4 α^*-closed map

Definition 4.1. A map $f : X \rightarrow Y$ is said to be α^*-closed map if for each closed set F in X, $f(F)$ is a α^*-closed set in Y.

Definition 4.2. A map $f : X \rightarrow Y$ is said to be α^*-open map if for each open set U in X, $f(U)$ is a α^*-open set in Y.

Definition 4.3. A map $f : X \rightarrow Y$ is said to be α^*-continuous map if $f^{-1}(V)$ is α^*-closed in X for each closed set V in Y.

Definition 4.4. A map $f : X \rightarrow Y$ is said to be a α^*-irresolute if $f^{-1}(V)$ is a α^*-closed in X for each α^*-closed set V in Y.

Lemma 4.1. *Every closed map is a α^*-closed map.*

Proof. Let $f : X \rightarrow Y$ be a closed map and let F be a closed set in X. Then, $f(F)$ is a closed set in Y and so α^*-closed Y. Thus, f is a α^*-closed map. □

The converse of the above Lemma need not be true in general.

Example 4.2. *Let $X = Y = \{a, b, c\}$ and let $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, c\}\}$ be topologies on X and Y respectively. Let $f(x) = x$ for every x in X. Then, f is a α^*-closed map. As the image of $\{c\}$ is not a closed set, f is not a closed map.*

Remark 4.1. Every g-closed map is a α^*-closed map.

Lemma 4.3. *If $f : X \rightarrow Y$ is a α^*-closed map and if $A = f^{-1}(B)$ for some closed set B in Y, then $f_A : A \rightarrow Y$ is a α^*-closed map.*

Proof. Let F be a closed set in A. Then, there is a closed set H in X such that $F = A \cap H$. Then,$f_A(F) = f(A \cap H) = f(A) \cap f(H) = B \cap f(H)$. Now $f(H)$ is a α^*-closed set in Y as f is a α^*-closed map. Therefore, $B \cap f(H)$ is a α^*-closed set in Y and so f_A is a α^*- closed map. □

Theorem 4.4. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be α^*-closed maps. If f is a closed map, then $gof : X \rightarrow Z$ is a α^*-closed map.*

Proof. Let F be a closed set in X. Then, $f(F)$ is a closed set in Y as f is a closed map. Then, $(g \circ f)(F) = g(f(F))$ is a α^*-closed set in Z as g is a α^*-closed map. Therefore, $g \circ f$ is a α^*-closed map. □

Lemma 4.5. *If $f : X \rightarrow Y$ is a α^*-irresolute, then f is a α^*-continuous map.*

Proof. Let F be any closed set in Y. Since every closed set is a α^*-closed set, F is a α^*-closed set in Y. Since f is a α^*-irresolute, $f^{-1}(F)$ is a α^*-closed set in X. Hence, f is a α^*-continuous. □

Lemma 4.6. *If $f : X \rightarrow Y$ is a α^*-continuous map and Y is a T_{α^*}-space, then f is a α^*-irresolute.*

Proof. Let F be a α^*-closed set in Y. Since Y is a T_{α^*}-space, F is a closed set. Then, $f^{-1}(F)$ is a α^*-closed set in X. Hence f is a α^*-irresolute. □

Theorem 4.7. *If $f : X \rightarrow Y$ is a α^*-irresolute and $g : Y \rightarrow Z$ is a α^*-continuous map, then $gof : X \rightarrow Z$ is a α^*-continuous map.*

Proof. Let F be a closed set in Z. Then, $g^{-1}(F)$ is a α^*-closed set in Y as g is α^*-continuous. Now $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is a α^*-closed set in X as f is a α^*-irresolute. Therefore, $g \circ f$ is a α^*-continuous map. □

Corollary 4.8. *If $f : X \to Y$ and $g : Y \to Z$ are α^*-continuous maps and Y is a T_{α^*}-space, then $g \circ f$ is a α^*-continuous map.*

Proof. In T_{α^*}-space, each α^*-closed set is a closed set, the result is directly follows from theorem 4.2. □

Lemma 4.9. *Every continuous map is a α^*-continuous map.*

Proof. Let $f : X \to Y$ be a continuous map and G be an open set in Y. Then, $f^{-1}(G)$ is an open set in X and hence α^*-open set in X. Therefore, f is a α^*-continuous map. □

Remark 4.2. Every g-continuous map is a α^*-continuous map.

5 α^*-Homeomorphism

Definition 5.1. A bijection $f : X \to Y$ is called α^*- homeomorphism when f is both α^*-continuous and α^*-closed map.

Lemma 5.1. *Every homeomorphism is a α^*-homeomorphism.*

Proof. Let $f : X \to Y$ be a homeomorphism. Then, f is both continuous and closed. Then, clearly f is a α^*-continuous and α^*-closed. So f is a α^*-homeomorphism. □

Lemma 5.2. *Every g-homeomorphism is a α^*-homeomorphism.*

Proof. Let $f : X \to Y$ be a g-homeomorphism. Then, f is both g-continuous and g-closed. Then, clearly f is α^*-continuous and α^*-closed. So f is a α^*-homeomorphism. □

The converse of the above two lemmas need not be true as seen from the following example.

Example 5.3. *Let X with a topology $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and Y with a topology $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$ where $X = Y = \{a, b, c\}$. If $f : X \to Y$ with $f(a) = a,\ f(b) = c$ and $f(c) = b$. Then, f is a α^*-homeomorphism, but not a homeomorphism and also not a g-homeomorphism as the inverse image of $\{a, b\}$ in Y is not closed and also not g-closed in X.*

Theorem 5.4. *For any bijection $f : X \to Y$, the following statements are equivalent:*

(a) the inverse map $f^{-1} : Y \to X$ is a α^*- continuous map,

(b) f is a α^*-open map,

(c) f is a α^*-closed map.

Proof. Let $f^{-1} : Y \to X$ be a α^*-continuous map and G be any open set in X. Then, the inverse image of G under f^{-1}, $f(G)$, is α^*-open in Y and so f is a α^*-open map. Now, let f be a α^*-open map and let F be any closed set in X. Then, F^c is open in X so $f(F^c)$ is α^*-open in Y. But $f(F^c) = Y \setminus f(F)$ and so $f(F)$ is α^*- closed in Y. Therefore, f is a α^*- closed map. Finally, let f be a α^*-closed map and let F be any closed set in X. Then, $f(F)$ is α^*-closed in Y. But $f(F)$ is the inverse image of F under f^{-1}. Therefore, f^{-1} is α^*- continuous. □

Theorem 5.5. *Let $f : X \to Y$ be a α^*- continuous map from a space X onto a space Y. Then, the following statements are equivalent:*

(a) *f is a α^*-open map,*

(b) *f is a α^*-homeomorphism,*

(c) *f is a α^*-closed map.*

Proof. Assume that f is a α^*-open map. Then, clearly f is a α^*- homeomorphism. Now, if f is a α^*-homeomorphism, then, by definition f is a α^*-closed map. Finally, if f is a α^*-closed map, then, by Theorem 5.3, f is a α^*-open map. □

The following example shows that, in general, the composition of two α^*-homeomorphisms need not be a α^*-homeomorphism.

Example 5.6. *Let $X = Y = Z = \{a, b, c\}$ be topologies with $\tau = \{X, \phi, \{a\}, \{a, b\}\}$, $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$ and $\eta = \{Z, \phi, \{a\}, \{b\}, \{a, b\}\}$ respectively. Let $f : X \to Y$ with $f(a) = a$, $f(b) = c$, $f(c) = b$ and let $g : Y \to Z$ with $g(x) = x$ for each x in Y. Then, both f and g are α^*-homeomorphisms, but their composition $gof : X \to Z$ is not a α^*- homeomorphism as $\{a, c\}$ is closed in Z, but $(gof)^{-1}(\{a, c\}) = \{a, b\}$ is not α^*-closed in Z.*

Theorem 5.7. *Let X and Z be any two topological spaces and let Y be a T_{α^*}-space. If $f : X \to Y$ and $g : Y \to Z$ be α^*-homeomorphisms, then the composition $g \circ f : X \to Z$ is a α^*-homeomorphism.*

Proof. Let F be a closed set in Z. Then, $g^{-1}(F)$ is a α^*-closed set in Y as g is a α^*-continuous map. Since Y is a T_{α^*}-space, $g^{-1}(F)$ is a closed set in Y. Thus $f^{-1}(g^{-1}(F))$=$(g \circ f)^{-1}(F)$ is a α^*-closed set in X. Thus $g \circ f$ is a α^*-continuous map.

Again, let F be a closed set in X. Then, $f(F)$ is a α^*-closed set in Y as f is a α^*-closed map. Since Y is a T_{α^*}-space, $f(F)$ is a closed set in Y. Thus $g(f(F)) = (g \circ f)(F)$ is a α^*-closed set in Z. Thus $g \circ f$ is a α^*-closed map. Hence $g \circ f$ is a α^*-homeomorphism. □

6 Conclusions

In this paper, we introduced a new kind of generalized closed sets, α^*-closed sets, and investigated their properties. The α^*-closed maps, α^*-continuous maps and α^*-irresolutes were also defined and investigated their properties. Finally, the α^*-homeomorphisms were introduced and their properties were established.

References

Levine, N. (1970). Generalized closed sets in Topology. Rend. Circ. Mat. Palermo, 19, 89-96.

Maki, H. Sundaram, P. and Balachandran, K. (1991). On generalized homeomorphisms in topological spaces. Bull. Fukuoka Univ. Ed. Part III, 40, 23-31.

Bhuvaneswari, K. and Mythili Gnanapriya, K. (2016). Nano Generalized Pre Homeomorphisms in Nano Topological Space. Bull. International Journal of Scientific and Research Publications, 6(7), 526-530.

Kadambavanam, K. and Vaithiyalingam, K. (2016). Weakly π Generalized Homeomorphism in Intuitionistic Fuzzy Topology. American Journal of Engineering Research (AJER), 5(10), 277-282.

Charanya, S. and Ramasamy, Dr. K. (2017). Pre Semi Homeomorphisms and Generalized Semi Pre Homeomorphisms in Topological Spaces. International Journal of Mathematics Trends and Technology (IJMTT), 42(1), 16-24.

Vivekananda Dembre, Pravin G Dhawale, and Devendra Gowda. (2018). New semi generalized spaces in topological spaces. International Journal of Applied Research, 4(2), 99-102.

Crossley, S. G. and Hilderbrand, S. K. (1972). Semi topoogical properties. Fund. Math., 74, 233-254.

Balachandran, K. Sundaram P. and Maki. H (1991). On generalized continuous maps in topological spaces. Mem. Fac. Sci. Kochi Univ. Ser.A. Math., 12, 5-13.

Andrijevic, D. (1986). Semi-preopen sets. Mat. Vesnik., 38(1), 24-32.