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Lotka-Volterra model assumes that the environment will always be homogeneous, but that is not the case since the environment is made up of many patches which are connected by migration [2], [8], [9], [13]. Migration occurs when a species moves from one patch to another due to some unfavorable conditions in its initial patch, for example, intraspecific competition, predation, overpopulation of either species in a patch, environmental factors like drought, human activities like logging among others, see for instance [3], [11]. Migration can either be constant or variable. For constant migration, the number of species moving per unit time is a constant fraction of the population of the species in a given patch while for variable migration, the number of species moving per unit time is dependent on the density of the other species in that patch and other factors.

The classical predator-prey models assume that in the absence of predation, the prey will grow in the Malthusian way (unboundedly) in the absence of the predator. This is not realistic because, apart from predation, other factors such as limited food resources, diseases and poaching may affect the prey population (as suggested by [3]). This necessitates an inclusion of a variable that will make the model to be bounded from above (the population not to explode), this variable is known as the carrying capacity. The carrying capacity is usually determined by the available sustaining resources. The resulting model is referred to as a Predator-Prey Model with logistic growth [9]. Examples of Predator-Prey Models with logistic growth are Verhulst Model [4], [9] and Rosenzweig and MacArthur Model [1], [7], [12].

Abdllaoui et. al [2], Comins and Blatt [5] and Mchich et. al [8] in their models assume that migration of these species is instantaneous, that is, the prey migrate immediately after a predator attack and the predator migrate after lacking their food source. In reality, this may not be the case since the species can meet different barriers like a swollen river, an infrastructure through the natural habitat or even decide to wait for their family to migrate together as it is seen when the wildebeest migrate. These barriers, therefore dictate the inclusion of a time delay in migration of these species [3], [10], [13].

2 The model

The proposed model uses the framework of the Rosenzweig-MacArthur predator-prey system with a Holling-type II predator functional response, where these two patches are coupled via migration, given by,

$$\begin{aligned}\dot{n}_1(t) &= \alpha(n_2(t - \tau) - n_1(t)) + n_1 \left(r_1 \left(1 - \frac{n_1}{K_1} \right) - \frac{A_1 p_1}{n_1 + B_1} \right), \\ \dot{p}_1(t) &= \beta(p_2(t - \tau) - p_1(t)) + \frac{p_1 n_1}{B_1 + n_1} - d_1 p_1\end{aligned}$$

$$\begin{aligned}\dot{n}_2(t) &= \alpha(n_1(t - \tau) - n_2(t)) + n_2 \left(r_2 \left(1 - \frac{n_2}{K_2} \right) - \frac{A_2 p_2}{n_2 + B_2} \right), \\ \dot{p}_2(t) &= \beta(p_1(t - \tau) - p_2(t)) + \frac{p_2 n_2}{B_2 + n_2} - d_2 p_2\end{aligned}\quad (1)$$

Where $n_i = n_i(t)$ and $p_i = p_i(t)$ are the prey and predator densities at time t in patch i , respectively. The parameter r_i denotes the prey intrinsic growth rate, The prey carrying capacity is denoted by K_i . The constant d_i is the natural mortality rate of the predator population. The predation parameter is denoted B_i and the prey capturing rate is denoted by A_i . The constant β denotes the predator migration rate, α is the prey migration rate and τ represents a time delay in the migration of both the prey and predator.

3 Model Analysis

Let $Z_i(t) := (n_i(t), p_i(t))$, $i = 1, 2$, then Equation (1) becomes,

$$\dot{Z}_i(t) = (Z_j(t - \tau) - Z_i(t))\gamma + f_i(Z_i(t)), \quad i \neq j, \quad (2)$$

where, $\gamma = (\alpha \quad \beta)^T$ and

$$f_i(Z_i(t)) = \begin{pmatrix} n_i \left(r_i \left(1 - \frac{n_i}{K_i} \right) - \frac{A_i p_i}{n_i + B_i} \right) \\ \frac{p_i n_i}{B_i + n_i} - d_i p_i \end{pmatrix}.$$

Let $Z(t) := (Z_1(t), Z_2(t))$ and $f(Z(t), Z(t - \tau))$ represent the vector field on the right hand side of Equation (2), thus Equation (2) becomes,

$$\dot{Z}(t) = f(Z(t), Z(t - \tau)). \quad (3)$$

Let $C = \mathcal{C}([-\tau, 0], \mathbb{R}^4)$ be a Banach space equipped with the sup norm, $\|\phi\| = \sup |\phi(\theta)| \leq r$, $(0 \leq r < \infty)$, for $\theta \in [-\tau, 0]$ and where $|\phi(\theta)|$ denotes a Euclidean norm of $\phi(\theta)$.

Let the initial condition be given by,

$$\varphi(t) := Z(t) |_{[-\tau, 0]}, \quad (4)$$

where $\varphi \in C$. Since $f(Z(t), Z(t - \tau)) \in \mathcal{C}(\mathbb{R}^4 \times C, \mathbb{R}^4)$, Equation (3) subject to Equation (4) has a unique solution. For more on existence and uniqueness of solutions, see for instance Hale & Lunel [6].

We wish to exploit the symmetries in the coupling terms of the linear part of Equation (2), that is;

$$\dot{Z}_i(t) = (Z_j(t - \tau) - Z_i(t))\gamma, \quad i, j = 1, 2, i \neq j, \quad (5)$$

which can be represented by two invariant manifolds. We will study the dynamics of these two invariant manifolds. To obtain these invariant manifolds, an application of the Laplace transform methods in complex variables to the terms describing migration in Equation (2) is required. For us to get the Laplace transform, an exponential estimate of the solution of the Equation (5) should be bounded.

3.1 Exponential Boundedness

We now prove that Equation (5) is exponentially bounded.

Lemma 3.1. *The solution of Equation (5) subject to the initial condition in Equation (4) for $t \geq 0$, satisfies,*

$$|Z(t)| \leq (1 + A\tau)e^{2At} |\varphi|, \quad (6)$$

where $|\cdot|$ denotes a sup norm in \mathbb{R} as well as a matrix norm.

Proof. Let $\begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix} = A_i$. Equation (5) can be written as,

$$\dot{Z}(t) = \begin{pmatrix} -A_1 & 0 \\ 0 & -A_2 \end{pmatrix} Z(t) + \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} Z(t - \tau).$$

The solutions of Equation (7) subject to initial condition in Equation (4) satisfy,

$$Z(t) = \varphi(0) + \int_{-\tau}^0 \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} \varphi(s) ds + \int_0^t \left\{ \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} Z(s - \tau) - \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} Z(s) \right\} ds.$$

Therefore,

$$\begin{aligned} |Z(t)| &\leq |\varphi| + A|\varphi|\tau + 2A \int_0^t |Z(s)| ds \\ &= (1 + A\tau) |\varphi| + 2 \int_0^t A |Z(s)| ds. \end{aligned}$$

Where $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = A$. Since $(1 + A\tau)\varphi$ is nondecreasing, by Grownwall's inequality,

$$|Z(t)| \leq (1 + A\tau) \exp\left(\int_0^t 2A ds\right) |\varphi| = (1 + A\tau) \exp(2At) |\varphi|.$$

□

Since Equation (5) is exponentially bounded, we will now find the characteristic equation so that we can be able to show that Laplace Transform of Equation (5) exists.

3.2 Characteristic Equation of the Coupling Terms

To obtain solutions of Equation (1), we assume a solution of the form

$$Z(t) = e^{\lambda t} c, \quad (7)$$

where $c \in \mathbb{R}^4$ with c a nonzero 4 by 1 column vector. Substituting Equations (7) into Equation (5), we obtain the characteristic equation

$$(\alpha + \lambda)^2(\beta + \lambda)^2 + \alpha^2\beta^2e^{-4\lambda\tau} - (\alpha + \lambda)^2\beta^2e^{-2\lambda\tau} - (\beta + \lambda)^2\alpha^2e^{-2\lambda\tau} = 0 \quad (8)$$

3.3 Invariant Manifold

We now show that solutions of Equation (5) define two semi-flows on two two-dimensional invariant subspaces of \mathbb{R}^4 . Taking the Laplace transform of Equation (5), we get

$$\begin{pmatrix} -\lambda I_2 - I_2\gamma & e^{-\lambda\tau} I_2\gamma \\ e^{-\lambda\tau} I_2\gamma & -\lambda I_2 - I_2\gamma \end{pmatrix} \begin{pmatrix} Z_1(\lambda) \\ Z_2(\lambda) \end{pmatrix} = \begin{pmatrix} Z_1(0) \\ Z_2(0) \end{pmatrix}. \quad (9)$$

where $\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Equation (9) is symmetric in nature. On simplifying equation (9), adding the set of equations involving $Z_2(0)$ to $Z_1(0)$ in equation (9), we obtain

$$(-I_2\lambda - I_2\gamma + I_2\gamma e^{-\lambda\tau})(Z_1(\lambda) + Z_2(\lambda)) = Z_1(0) + Z_2(0), \quad (10)$$

On simplifying equation (9) by subtracting the set of equations involving $Z_2(0)$ from $Z_1(0)$ in equation (9), we obtain

$$(-I_2\lambda - I_2\gamma - I_2\gamma e^{-\lambda\tau})(Z_1(\lambda) - Z_2(\lambda)) = Z_1(0) - Z_2(0), \quad (11)$$

The Matrix $(-I_2\lambda - I_2\gamma + I_2\gamma e^{-\lambda\tau})^{-1}$ is non-singular when λ is such that $(-I_2\lambda - \gamma + \gamma e^{-\lambda\tau}) \neq 0$ and $(-\lambda - \gamma - \gamma e^{-\lambda\tau})^{-1} I_2$ is non-singular when λ is such that $(-\lambda - \gamma - \gamma e^{-\lambda\tau}) \neq 0$, thus the inverse Laplace transform for Equation (10) and Equation (11) is

$$\begin{aligned} (Z_1(\lambda) + Z_2(\lambda)) &= \mathcal{L}^{-1}\{(-\lambda - \gamma + \gamma e^{-\lambda\tau})^{-1}\}(Z_1(0) + Z_2(0)), \\ (Z_1(\lambda) - Z_2(\lambda)) &= \mathcal{L}^{-1}\{(-\lambda - \gamma - \gamma e^{-\lambda\tau})^{-1}\}(Z_1(0) - Z_2(0)), \end{aligned} \quad (12)$$

We have two manifolds; the symmetric manifold denoted by Θ -manifold where $Z_1(t) = Z_2(t)$ and the asymmetric manifold denoted by Π -manifold where $Z_1(t) = -Z_2(t)$.

Let us now define two linear subspaces of \mathbb{R}^4

$$\begin{aligned}\Theta &= \{(Z_1(t), Z_2(t)), Z_i(t) \in \mathbb{R}^2 : Z_1(t) - Z_2(t) = 0\}, \\ \Pi &= \{(Z_1(t), Z_2(t)), Z_i(t) \in \mathbb{R}^2 : Z_1(t) + Z_2(t) = 0\}.\end{aligned}\quad (13)$$

To simplify the study of Equation (1) on these manifolds, we introduce a change of coordinates defined by,

$$\begin{aligned}u_1 &:= \frac{1}{2}(n_1 + n_2), \quad v_1 := \frac{1}{2}(p_1 + p_2), \\ u_2 &:= \frac{1}{2}(n_1 - n_2), \quad v_2 := \frac{1}{2}(p_1 - p_2).\end{aligned}\quad (14)$$

With the assumption that the predator and prey species are of the same type regardless of the patch, we take $r_1 = r_2 := r$, $A_1 = A_2 := A$, $B_1 = B_2 := B$, $d_1 = d_2 := d$, $\alpha_1 = \alpha_2 := \alpha$, $K_1 = K_2 := K$ and $\beta_1 = \beta_2 := \beta$. Using the transformation in Equation (14) in Equation (1), we obtain,

$$\begin{aligned}\dot{u}_1 &= \alpha(u_1(t - \tau) - u_1) + \frac{r}{K}(Ku_1 - u_1^2 - u_2^2) - A\left(\frac{Bu_1v_1 + u_1^2v_1 - u_2(-u_2v_1 + Bv_2)}{(u_1 + B)^2 - u_2^2}\right), \\ \dot{v}_1 &= \beta(v_1(t - \tau) - v_1) + \frac{Bu_2v_2 + (u_1(B + u_1) - u_2^2)v_1}{(B + u_1)^2 - u_2^2} - dv_1, \\ \dot{u}_2 &= -\alpha(u_2(t - \tau) + u_2) + \frac{r}{K}(Ku_2 - 2u_1u_2) - A\left(\frac{Bu_2v_1 + u_1(B + u_1)v_2 - u_2^2v_2}{(u_1 + B)^2 - u_2^2}\right), \\ \dot{v}_2 &= -\beta(v_2(t - \tau) + v_2) + \frac{Bu_2v_1 + (u_1(B + u_1) - u_2^2)v_2}{(B + u_1)^2 - u_2^2} - dv_2.\end{aligned}\quad (15)$$

where $u_i = u_i(t)$ and $v_i = v_i(t)$, $i = 1, 2$. The linear subspace in Equation (13) becomes

$$\begin{aligned}\Pi &= \{(u_1, v_1, 0, 0) \in \mathbb{R}^4 : (u_1, v_1) \in \mathbb{R}^2\}, \\ \Theta &= \{(0, 0, u_2, v_2) \in \mathbb{R}^4 : (u_2, v_2) \in \mathbb{R}^2\}.\end{aligned}$$

On both Π and Θ the system reduces to two dimensional systems of the form

$$\dot{U}_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} U_1(t - \tau) + \begin{pmatrix} -\alpha + r - \frac{ru_1}{K} & \frac{v_1}{u_1 + B} \\ 0 & -\beta - d \end{pmatrix} U_1 \quad (16)$$

and

$$\dot{U}_2 = \begin{pmatrix} -\alpha & 0 \\ 0 & -\beta \end{pmatrix} U_2(t - \tau) + \begin{pmatrix} -\alpha + r & \frac{u_2}{B^2 - u_2^2} \\ 0 & -\beta - d \end{pmatrix} U_2 \quad (17)$$

respectively. $U_i = (u_i, v_i)^T$

Next we examine the stability of solutions on the two manifolds, this will help us predict long-term behaviors of solutions of Model (1).

3.4 Asymmetric Manifold

On solving the system in equation (16), we let $U_1(t) = e^{\lambda t}C_1$, then we obtain the following characteristic equation,

$$(\alpha e^{-\lambda\tau} - \alpha + r - \lambda)(\beta e^{-\lambda\tau} - \beta - d - \lambda) = 0 \quad (18)$$

We will use the following theorem to show the nature of solutions for Equation(16)

Theorem 3.1. *For all β and s , equation (16) has*

- (i) *a sink at the origin for $\alpha > r$,*
- (ii) *a saddle at the origin for $\alpha < r$,*
- (iii) *a periodic solution for $\alpha = r$*

Proof. Using the first factor of equation (18), we have

$$(\alpha e^{-\lambda\tau} - \alpha + r - \lambda) = 0 \quad (19)$$

Let

$$z = (\lambda + \alpha - r)\tau \quad (20)$$

Then equation (19) becomes

$$z = \alpha\tau e^{-z} e^{(-r+\alpha)\tau} \quad (21)$$

The following lemma will be used to simplify equation (21)

Lemma 3.2. *The equation $z = be^{-z}$ has simple pure imaginary roots,*

$$z = i(\pi/2 + 2m\pi), \quad \text{for } b = -(\pi/2 + 2m\pi)$$

$$z = 0, \quad \text{for } b = 0$$

$$z = i(\pi/2 + (2m + 1)\pi), \quad \text{for } b = (\pi/2 + (2m + 1)\pi)$$

where $m = 0, 1, 2, \dots$ and there are no other purely imaginary roots.

Using Lemma (3.2) where $b = \alpha\tau e^{-(r-\alpha)\tau} > 0$, we get $z = i(\pi/2 + (2m+1)\pi)$ for $\alpha\tau e^{-(r-\alpha)\tau} = (\pi/2 + (2m + 1)\pi)$. Equation (20) becomes

$$\lambda = \frac{i(\pi/2 + (2m + 1)\pi)}{\tau} - \alpha + r \quad (22)$$

Equation (22) has

- (i) roots with positive real parts for $\alpha < r$
- (ii) roots with negative real parts for $\alpha > r$

(iii) purely imaginary roots for $\alpha = r$.

For the second factor of equation (18),

$$(\beta e^{-\lambda\tau} - \beta - d - \lambda) = 0. \quad (23)$$

Let

$$z = (\lambda + \beta + d)\tau \quad (24)$$

From Lemma (3.2), equation (24) becomes

$$\lambda = \frac{i(\pi/2 + (2m + 1)\pi)}{\tau} - \beta - d \quad (25)$$

Equation (24) has roots with negative real parts for all positive β and k . The results for equation (18) therefore becomes

- (i) a saddle at the origin for $\alpha < r$, meaning that, when the prey migration rate is less than the prey growth rate, then the prey population becomes extinct, due to the fact that the prey do not migrate in large quantities after a predator attack leaving a big fraction of the prey density vulnerable to other predator attacks and thus diminishing the prey density to extinction after some time. This will lead to the extinction of the predator population after some time due to their lack of food.
- (ii) a sink at the origin for $\alpha > r$, implying that, when the prey migration rate is greater than the prey growth rate, the two species will coexist. This is as a result of the prey migrating in large densities after a predator attack thus reducing the danger the prey density faces.
- (iii) a center for $\alpha = r$, that means that, when the prey migration rate is the same as the prey growth rate and the predator migration rate is equal to the predator carrying capacity, then a periodic solution for these species occurs. The prey and predator densities will be dependent on the other species, the prey population is governed by the availability of sustainable resources and the predator density in a given patch while the predator species is dependent on the availability of their food source.

□

3.5 Numerical analysis for Asymmetric manifold

The following parameter values, adapted from Apima [3], are used in simulating the graphs of equation (16), $K = 250$, $A = 0.3$, $B = 3$, $d = 0.4$, $\beta = 2.792403$, $u_1 = 240$ and $v_1 = 80$. Simulations for the asymmetric manifold gives,

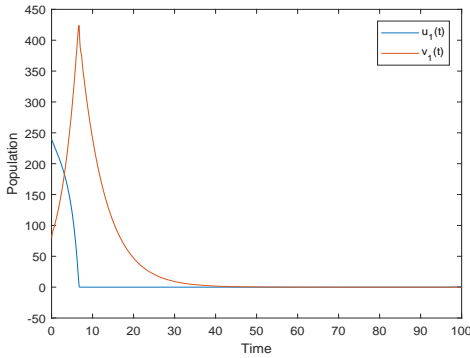


Figure 1: $\alpha < r$, $\alpha = 0.1$, $r = 0.6$, $\tau = 0.1$.

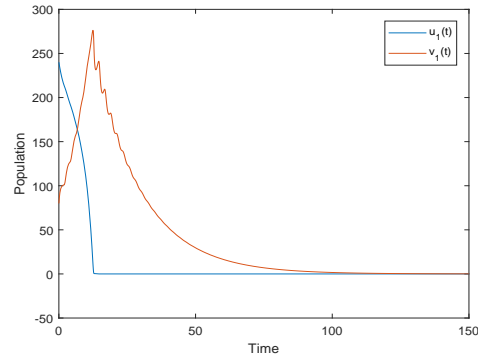


Figure 2: $\alpha < r$, $\alpha = 0.1$, $r = 0.6$, $\tau = 2$.

When the prey migration rate is less than the intrinsic growth rate ($\alpha < r$), the prey density becomes extinct after some time as shown in Figure 1 and Figure 2. This leads to the predator density also becoming extinct since it does not have any source of food. A longer delay makes the predator and prey species become extinct at a slower rate compared to when the delay is short. This is due to the fact that the prey species migrates at a slower rate, meaning few prey migrate, and those migrating face barriers which affect their migration and thus they remain the given patch for some more time and this in turn provides food for the predator density.

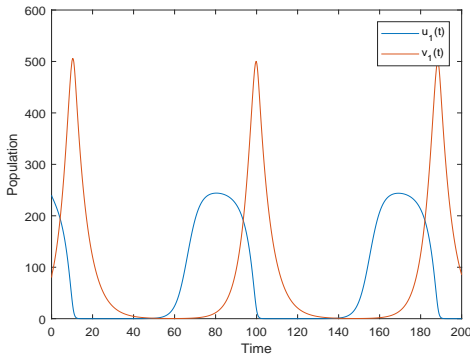


Figure 3: $\alpha > r$, $\alpha = 1.5$, $r = 0.6$, $\tau = 0.1$.

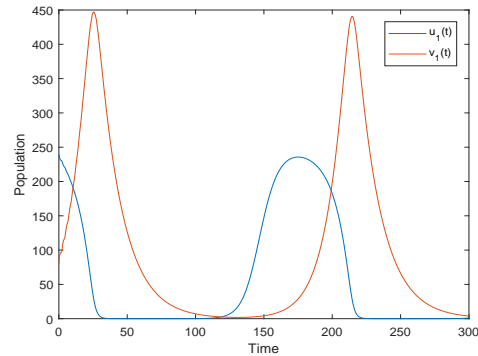


Figure 4: $\alpha > r$, $\alpha = 1.5$, $r = 0.6$, $\tau = 2$.

In Figure 3 and Figure 4, the prey migration rate is greater than the intrinsic growth rate ($\alpha > r$). The predator and prey densities coexist. These populations oscillates, where both densities are dependant on the available sustaining resources. A longer delay slows down the decrease or increase of a given species.

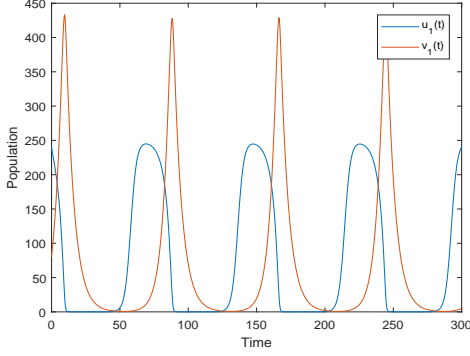


Figure 5: $\alpha = r$, $\alpha = r = 0.6$, $\tau = 0.1$.

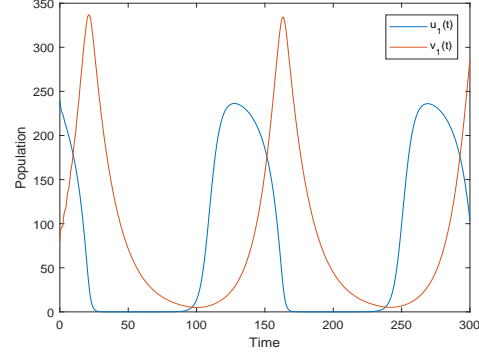


Figure 6: $\alpha = r$, $\alpha = r = 0.6$, $\tau = 10$.

In Figure 5 and Figure 6, the prey migration rate is equal to the intrinsic growth rate ($\alpha = r$). The predator and prey densities will oscillate. The prey density will depend on the availability of food and its safety from predation while the predator density will depend on the availability of the food source. A longer delay slows down the change in population of a given species.

3.6 Symmetric Manifold

Similarly, on solving the system in equation (17), let $U_2(t) = e^{\lambda t}C_2$, then the following characteristic equation is obtained,

$$(-\alpha e^{-\lambda\tau} - \alpha + r - \lambda)(-\beta e^{-\lambda\tau} - \beta - d - \lambda) = 0 \quad (26)$$

The following theorem is used to show the behaviour of solutions for equation (17),

Theorem 3.2. *For all values of β and s , equation (17) has*

- (i) *a sink at the origin for $\alpha > r$,*
- (ii) *a saddle at the origin for $\alpha < r$,*
- (iii) *a periodic solution $\alpha = r$*

Proof. Using the first factor of equation (26), we have

$$(-\alpha e^{-\lambda\tau} - \alpha + r - \lambda) = 0 \quad (27)$$

Let

$$z = (\lambda + \alpha - r)\tau \quad (28)$$

Then equation (27) becomes

$$z = -\alpha\tau e^{-z} e^{(-r+\alpha)\tau} \quad (29)$$

Using Lemma (3.2) where $b = -\alpha\tau e^{(-r+\alpha)\tau} > 0$, we get $z = i(\pi/2 + 2m\pi)$ for $-\alpha\tau e^{(-r+\alpha)\tau} = -(\pi/2 + 2m\pi)$. Equation (28) becomes

$$\lambda = \frac{i(\pi/2 + 2m\pi)}{\tau} - \alpha + r \quad (30)$$

Equation (30) has

- (i) roots with positive real parts for $\alpha < r$
- (ii) roots with negative real parts for $\alpha > r$
- (iii) purely imaginary roots for $\alpha = r$.

For the second factor of equation (26),

$$(-\beta e^{-\lambda\tau} - \beta - d - \lambda) = 0. \quad (31)$$

Let

$$z = (\lambda + \beta + d)\tau \quad (32)$$

From Lemma (3.2), equation (32) becomes

$$\lambda = \frac{i(\pi/2 + 2m\pi)}{\tau} - \beta - d \quad (33)$$

Equation (33) has roots with negative real parts for all positive β and k . The results for equation (26) therefore becomes

- (i) a saddle at the origin for $\alpha < r$, meaning that, when the prey migration rate is less than the prey growth rate, then model becomes unstable. Therefore, the prey population becomes extinct which makes the predator population to be wiped out after some time due to their lack of food.
- (ii) a sink at the origin for $\alpha > r$, implying that, when the prey migration rate is greater than the prey growth rate, then the model is stable and therefore the two species will coexist.
- (iii) a center for either $\alpha = r$, meaning that, when the prey migration rate is the same as the prey growth rate, a periodic solution occurs.

□

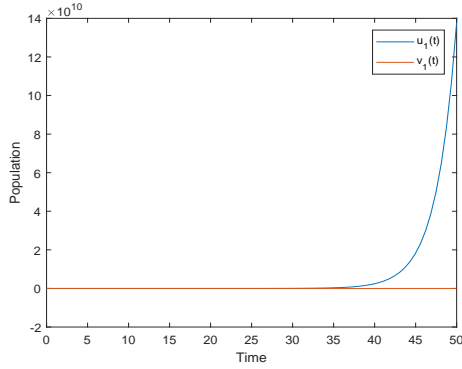


Figure 7: $\alpha < r$, $\alpha = 0.1$, $r = 0.6$, $\tau = 0.1$.

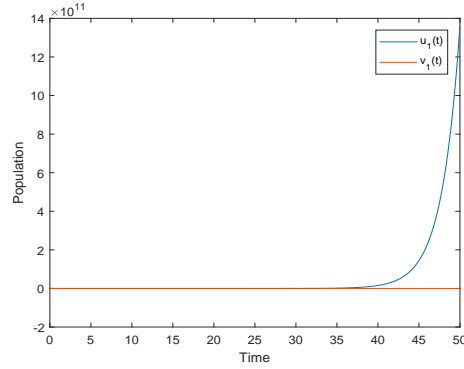


Figure 8: $\alpha < r$, $\alpha = 0.1$, $r = 0.6$, $\tau = 1.5$.

3.7 Numerical analysis for Symmetric manifold

The following parameter values, adapted from Apima [3], are used in simulating the results of equation (17), $\beta = 2.792403$, $B = 3$, $d = 0.4$, $u_2 = 240$ and $v_2 = 80$. Simulations for the Symmetric manifold gives,

When the prey migration rate is less than the intrinsic growth rate ($\alpha < r$), the predator density becomes extinct leading to the prey density growing exponentially. This is an unstable case. A longer delay makes the population increase as seen in Figure 8 compared to Figure 7.

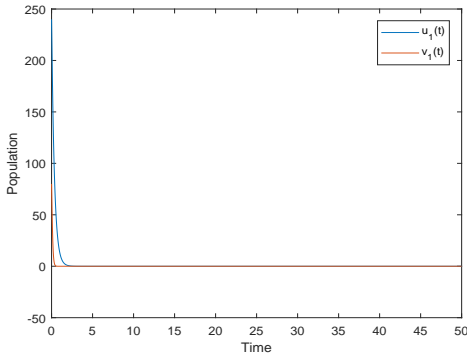


Figure 9: $\alpha > r$, $\alpha = 1.5$, $r = 0.6$, $\tau = 0.1$.

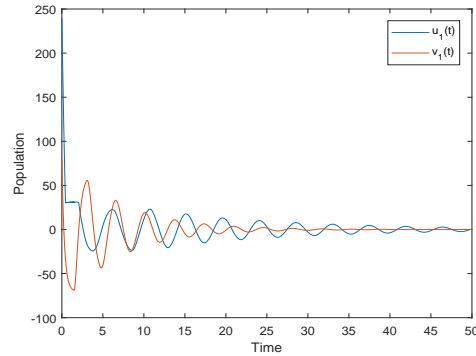


Figure 10: $\alpha > r$, $\alpha = 1.5$, $r = 0.6$, $\tau = 1.5$.

When prey migration rate is greater than the intrinsic growth rate ($\alpha > r$), the two densities stabilize at zero as seen in Figure 9. Implying that the prey and predator densities in the two patches are equal (using equation (14)). With a longer delay, the populations oscillate as seen in Figure 10, when the population is positive, then the population in patch one is greater than the population in patch two and when the population is negative, then the population in patch

one is less than the population in patch two.

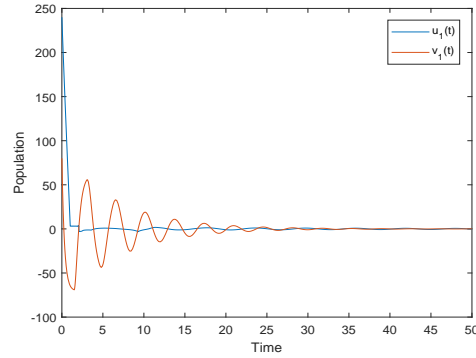
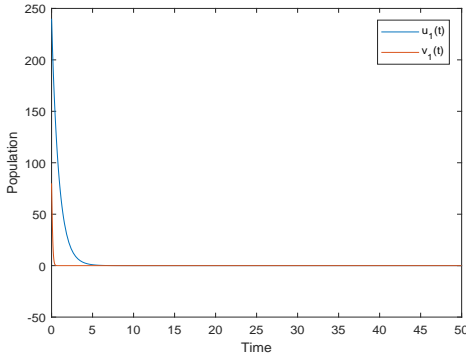


Figure 11: $\alpha = r$, $\alpha = 0.6$, $r = 0.6$, $\tau = 0.1$. Figure 12: $\alpha = r$, $\alpha = 0.6$, $r = 0.6$, $\tau = 1.5$.

In Figure 11 and Figure 12, the prey migration rate is equal to the intrinsic growth rate ($\alpha = r$). The two densities stabilize at zero after some time. A longer delay introduces oscillations and the time taken for the model to stabilize in this case increases.

4 Conclusion

These results show that migration plays a crucial role in the existence of the predator and prey species. The results also show that when we have a longer delay in migration for the species, the model stabilize at a slower rate compared to when the delay is short. The relevant agencies should address factors that affect the rate of migration of the predator and prey species, for example, minimizing human settlement and activities in the natural habitats of these species which are barriers to migration.

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