THE SOLUTION OF LINEARISED KORTEweg-De Vries Equation Using THE DIFFERENTIAL TRANSFORM METHOD

ABSTRACT

This paper considers the Differential Transform Method for solving partial differential equations. The two-dimensional form of the method is used to solve a linearised version of the Korteweg-de Vries equation. The results show that the differential transform method is both a highly effective method and also a very simple method to apply.

Keywords: The Korteweg-de Vries equation, the Differential Transform Method.

1 Introduction

Partial differential equations (PDEs) describe in the best possible manner the dynamical behaviour of the various systems found in many branches of the physical sciences. Several classical methods have been devised in the past decades to solve these equations. Some of these methods are the Adomian Decomposition Method ([36]), the Petro-Galerkin method ([45]), the Double-Sub-Equation method ([13]), the (G'/G)-Expansion method ([4]) the G'/G-Expansion method ([41]), the Variational Iteration method ([37]), the Homotopy Perturbation method ([2], [16]), the Exp-function method ([3]) and the Differential Transform Method (DTM), which is considered in this paper.

The DTM is an anaytical method that generates solutions in the form of a Taylor series. The method employs an iterative procedure in its application. It was Chen ([12]) who first proposed the DTM and it was used to solve linear and nonlinear initial value problems in electrical circuit analysis. Since then, the method has been used to solve a wide range of equations. These include the Poisson, Wave and Heat equations ([25]), the Volterra-integral equations ([29]), the Burgers’equation ([30]), the Zakharov-Kuznetsov and Helmholtz equations ([38]), the quadratic Riccati differential equations
The telegraph equation ([8], [39]), the Lienard equation ([28]), the Volterra-Fredholm integro-differential equation ([7]), complex differential equation systems ([17]) and a predator-prey problem ([6]). The DTM has found widespread commendation for its accuracy, computational efficiency and ease of application when compared with other methods ([5], [34], [35]).

In this paper the DTM is employed to solve a linearised Korteweg-de Vries (KdV) equation, which is a special case of the generalised KdV equation. The KdV equation is a nonlinear dispersive partial differential equation which describes solitary water waves (also called solitons) in a shallow water domain ([11], [26], [43]). Since soliton excitations form the basis of several phenomena in a variety of fields such as plasma physics, geophysics, fibre optics, and others, the KdV equation plays a fundamental role in many areas of study ([10], [31], [32]). A wide range of applications of the KdV equation are discussed by many authors, see for example [1], [14], [18], [19], [20], [21], [22], [24], [27], [42], and [44].

Several variations of the KdV equation have been studied in literature. These include the modified KdV equation ([10]), the complex modified KdV equation ([33]), the fifth order KdV equation ([27]), and the linearised KdV equation ([15], [40], [43], [45]) which is studied in this paper. Zhang ([45]) investigated numerical solutions of the linearised KdV equation with absorbing boundary conditions using the dual-Petrov-Galerkin method. Sobirov ([40]) investigated the linearised KdV equation on a metric star graph with three semi-infinite bonds connected at a vertex. Deconinck ([15]) used the Fokas’ unified transform method to construct a solution of the linearised KdV equation and provided an explicit characterisation of the sufficient interface conditions.

Villanueva ([43]) used a special ansatz substitution to study the linearised KdV equation with time dependent coefficients in the form:

$$u_t = -a(t)u_{ttt} + b(t)u_x + c(t)u,$$  \hspace{1cm} (1.1)

where \(a(t), b(t),\) and \(c(t)\) are integrable functions of \(t\). Two examples were solved by Villanueva ([43]), one in which \(a(t) = 1, b(t) = c(t) = 0\), and the other in which \(a(t) = \cos t, b(t) = 0\) and \(c(t) = 3t^2\). In this paper the Differential Transform Method is employed to solve equation (1.1) where \(a(t), b(t)\) and \(c(t)\) are constants and initial values are prescribed.

## 2 The one-dimensional differential transform

Following the approach by Hussin ([23]), if a function \(u(x)\) is continuously differentiable on the interval \((x_0 - \epsilon, x_0 + \epsilon)\) for \(\epsilon > 0\), then the differential transform for the \(k^{th}\) derivative of \(u(x)\) is defined as:

$$U(k) = \frac{1}{k!} \left[ \frac{d^k u(x)}{dx^k} \right]_{x=x_0},$$  \hspace{1cm} (2.1)

where \(U(k)\) is the transformed function or the \(T\)-function. The inverse differential transform of \(U(k)\) is defined as

$$u(x) = \sum_{k=0}^{\infty} U(k) (x-x_0)^k.$$  \hspace{1cm} (2.2)

When (2.1) is substituted in (2.2), the inverse transform becomes

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \frac{d^k u(x)}{dx^k} \right]_{x=x_0} (x-x_0)^k.$$  \hspace{1cm} (2.3)

The inverse differential transform given by (2.3) is the same as the Taylor series expansion of \(u(x)\) about \(x = x_0\).
From definitions (2.2) and (2.3) the following theorems of the differential transforms follow (Hussin ([23])).

Theorem 2.1. If \( u(x) = f(x) \pm g(x) \), then \( U(k) = F(k) \pm G(k) \).

Theorem 2.2. If \( u(x) = cf(x) \) where \( c \) is a constant, \( U(k) = cF(k) \).

Theorem 2.3. If \( u(x) = \frac{df(x)}{dx} \), then \( U(k) = (k + 1)F(k + 1) \).

Theorem 2.4. If \( u(x) = \frac{d^2f(x)}{dx^2} \), then \( U(k) = (k + 1)(k + 2)F(k + 2) \).

Theorem 2.5. If \( u(x) = \frac{d^rf(x)}{dx^r} \), then \( U(k) = (k + 1)(k + 2)\cdots(k + r)F(k + r) = \frac{(k + r)!}{k!}F(k + r) \).

Note that Theorem 2.5 is a generalisation of Theorems 2.3 and 2.4.

Theorem 2.6. If \( u(x) = f(x)g(x) \), then \( U(k) = \sum_{r=0}^{k} F(r)G(k - r) = \sum_{r=0}^{k} F(k - r)G(r) \).

Theorem 2.7. If \( u(x) = x^m \), where \( m \) is a positive integer, then \( U(k) = \delta(k - m) \), where \( \delta(k - m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases} \)

Theorem 2.8. If \( u(x) = e^{\alpha x} \), then \( U(k) = \frac{\alpha^k}{k!} \).

Theorem 2.9. If \( u(x) = \sin(\alpha x + \beta) \), then \( U(k) = \frac{\alpha^k}{k!} \sin \left( \frac{\pi k}{2} + \beta \right) \).

Theorem 2.10. If \( u(x) = \cos(\alpha x + \beta) \), then \( U(k) = \frac{\alpha^k}{k!} \cos \left( \frac{\pi k}{2} + \beta \right) \).

3 The two-dimensional differential transform

In two dimensions, the differential transform of a function \( w(x,y) \) at the point \((x_0, y_0)\) may be defined as follows ([12]).

\[
W(k,h) = \frac{1}{k!h!} \left[ \frac{\partial^{k+h}w(x,y)}{\partial x^k \partial y^h} \right]_{x=x_0, y=y_0},
\]

(3.1)

where \( W(k,h) \) is the transformed function or the \( T \)-function. The inverse differential transform of \( W(k,h) \) is defined as

\[
w(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k,h)(x - x_0)^k (y - y_0)^h.
\]

(3.2)

Substituting (3.1) in (3.2), the complete form of the inverse differential transform is given by

\[
w(x,y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \left[ \frac{\partial^{k+h}w(x,y)}{\partial x^k \partial y^h} \right]_{x=x_0, y=y_0} (x - x_0)^k (y - y_0)^h.
\]

(3.3)
Now, expression (3.3) is the Taylor series expansion of \(w(x, y)\) about \((x_0, y_0)\).

For practical applications of the differential transform method, it is much more convenient to consider \(w(x, y)\) and \(W(k, h)\) at \((0, 0)\) rather than at \((x_0, y_0)\). In this case, the differential transform \(W(k, h)\) may be written as

\[
W(k, h) = \frac{1}{k!h!} \left[ \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} \right]_{x=0, y=0},
\]

(3.4)

and the corresponding inverse differential transform is given by

\[
w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!h!} \left[ \frac{\partial^{k+h} w(x, y)}{\partial x^k \partial y^h} \right]_{x=0, y=0} x^k y^h.
\]

(3.5)

The following theorems can be easily deduced from the definitions (3.4) and (3.5).

Theorem 3.1. If \(w(x, y) = u(x, y) \pm v(x, y)\), then \(W(k, h) = U(k, h) \pm V(k, h)\).

Theorem 3.2. If \(w(x, y) = cu(x, y)\), then \(W(k, h) = cU(k, h)\), where \(c\) is a constant.

Theorem 3.3. If \(w(x, y) = \frac{\partial u(x, y)}{\partial x}\), then \(W(k, h) = (k + 1)U(k + 1, h)\).

Theorem 3.4. If \(w(x, y) = \frac{\partial v(x, y)}{\partial y}\), then \(W(k, h) = (h + 1)V(k, h + 1)\).

Theorem 3.5. If \(w(x, y) = \frac{\partial^{r+s} u(x, y)}{\partial x^r \partial y^s}\), then
\[
W(k, h) = \frac{(k + 1)(k + 2)\cdots(k + r)(h + 1)(h + 2)\cdots(h + s)}{(k + r)!(h + s)!} U(k + r, h + s),
\]

(3.6)

Theorem 3.6. If \(w(x, y) = u(x, y)v(x, y)\), then
\[
W(k, h) = \sum_{r=0}^{k} \sum_{s=0}^{h} U(r, h - s)V(k - r, s),
\]

(3.7)

\[
= \sum_{r=0}^{k} \sum_{s=0}^{h} U(k - r, s)V(r, h - s).
\]

Theorem 3.7. If \(w(x, y) = x^m y^n\), then \(W(k, h) = \delta(k - m, h - n) = \delta(k - m)\delta(h - n)\),

where \(\delta(k - m) = \begin{cases} 1, & k = m, \\ 0, & k \neq m, \end{cases}\) and \(\delta(h - n) = \begin{cases} 1, & h = n, \\ 0, & h \neq n. \end{cases}\)

4 Application to the linearised KdV equation

The nonlinear KdV equation, in a simplified form, is given by ([43])

\[
ux + u_{xxx} + 6uu_x = 0,
\]

(4.1)

where \(u(x, t)\) denotes the displacement of water (or any other medium) from its equilibrium depth at location \(x\) and at a time \(t\). Various versions of the linearised form of (4.1) have been studied in literature. Solutions of some forms have been investigated by Denoninck et. al. ([15]), Sobirov et. al. ([40]), Villanueva ([43]) and Zheng et. al. ([45]). This paper considers the solution of two initial value problems of the linearised KdV equation using the DTM approach.
4.1 Example 4.1
Consider the linearised KdV equation given by
\[ u_t + u_x + 6u_{xxx} = 0, \]  
subject to the initial condition
\[ u(x, 0) = \sin 2x. \]

The differential transform of (4.2) is given by
\[ (h + 1)U(k, h + 1) + (k + 1)U(k + 1, h) + 6(k + 1)(k + 2)(k + 3)U(k + 3, h) = 0. \]

Replacing \( h \) by \( h - 1 \) followed by a rearrangement of terms, (4.4) becomes
\[ U(k, h) = -\frac{(k + 1)}{h} U(k + 1, h - 1) - \frac{6(k + 1)(k + 2)(k + 3)}{h} U(k + 3, h - 1). \]

The initial condition (4.3) is a function of \( x \) alone, so that \( u(x, 0) \) may be treated as \( u(x) \) for the purpose of determining its differential transform. Applying Theorem 2.9 to (4.3) gives
\[ U(k, 0) = \frac{2k}{k!} \sin \left( \frac{k\pi}{2} \right). \]

The recurrence relation (4.5) together with (4.6) gives
\[ U(k, h) = \frac{46h^2 k}{k! h!} \sin \left( \frac{(k + h)\pi}{2} \right), \quad k, h = 0, 1, 2, 3, \ldots. \]

This is the recurrence relation that is used to determine the coefficients \( U(k, h) \) in the inverse transform given by (3.5). Substituting (4.7) into equation (3.5) yields, through successive simplifications,
\[ u(x, t) = \sin(2x + 46t). \]

It can be easily verified that (4.8) is a solution of the linearised KdV equation (4.2) with initial condition (4.3).

4.2 Example 4.2
In this example, \( u_x \) is omitted in equation (4.2) so that we consider the equation
\[ u_t = c^2 u_{xxx}, \]
subject to the initial condition
\[ u(x, 0) = \cos(ax), \]
where \( a \) and \( c \) are constants. The differential transform of (4.9) is given by
\[ (h + 1)U(k, h + 1) = c^2(k + 1)(k + 2)(k + 3)U(k + 3, h). \]

As in Example 1, upon replacing \( h \) by \( h - 1 \) and rearranging, (4.11) becomes
\[ U(k, h) = \frac{c^2(k + 1)(k + 2)(k + 3)}{h} U(k + 3, h - 1). \]

Applying Theorem 2.9 to the initial condition (4.10) gives
\[ U(k, 0) = \frac{a^k}{k!} \cos \left( \frac{k\pi}{2} \right). \]

5
Considering the recurrence relation (4.12) together with (4.13), the general formula for \( U(k, h) \) is given by

\[
U(k, h) = \left( \frac{(c^2)h}{k!} \right)^{a+h} \cos \left( \frac{(k + 3h)\pi}{2} \right), \quad k, h = 0, 1, 2, \ldots
\]  

(4.14)

After substituting (4.14) in (3.5) and a sequence of simplifications, the solution is obtained in the form

\[
u(x, t) = \cos(\alpha x - c^2 \alpha^3 t).
\]  

(4.15)

(4.15) satisfies the pair (4.9) and (4.10).

5 Conclusion

In this paper, the DTM approach has been used to solve two forms of the linearised KdV equation. The method is commended for its accuracy, computational efficiency and ease of application when compared with other methods. It does not require prior linearisation, discretisation or perturbation; in fact the solutions obtained in this paper are in closed form, which makes the method attractive to apply.

Competing Interests

The authors declare that no competing interests exist.

References


