A NEW AND SIMPLE PRIME SIEVING TECHNIQUE FOR GENERATING PRIME NUMBERS ENDING WITH A GIVEN ODD DIGIT

Abstract. The essence of this paper is to furnish a simple prime sieving technique which deletes composites from a finite list of natural numbers ending with any given odd digit with the exception of the digit 5, leaving behind prime numbers ending with the given digit. This technique is so much like the Eratosthenes' sieving technique.

1. Introduction

Prime numbers, as we know them, are highly mysterious and their origins are past finding out. In what pattern and order do the prime numbers arise in the sequence of natural numbers? [3] This question is among the most exquisite and ancient puzzles in the theory of numbers, and the answer to it has remained elusive to generations of mathematicians [4], [1].

For centuries, mathematicians, both great and little, have been searching for a formula for generating only prime numbers [2]. In the 17th Century, Pierre De Fermat (1601-1665) surmised that the formula $2^{2^n}+1$ would generate a prime for any whole number value of n. The first five numbers produced with this formula are all primes and are known as Fermat primes. In 1732, Leonhard Euler, however, found that $2^{2^5}+1 = 6416700417$ is composite. In 1880, Landry proved that $2^{2^6}+1 = 274177 \times 67280421310721310721$ is also composite. Today, no more Fermat primes have been found.

Worthy of note are two polynomial functions that generate prime numbers. In 1732, Euler gave to the world the polynomial function $f(n) = n^2 - n + 41$, which produces primes for n up to 40 and fails at $n = 41$. In 1879, E.B. Escott instituted the function $f(n) = n^2 - 79n + 1601$ to generate more primes, but this fails at $n = 80$. No function $f(n)$ which assumes all prime values and only prime values is known.

No efficient formula for finding the nth prime number exists. The Author's formula

$$P_{m+2} = \frac{1}{2}(6m - (-1)^m + 3), \quad m > 0$$

which generates the seven consecutive primes $P_3 = 5, P_4 = 7, P_5 = 11, P_6 = 13, P_7 = 17, P_8 = 19$, and $P_9 = 23$ fails when $m = 8$.

In the *Elements* of the great Euclid, there is an enchanting technique (but not a formula) for fishing out all and only primes from a list of natural numbers:

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$1, 2, 3, \ldots, n$. It is called the Sieve of Eratosthenes, named after its inventor the illustrious Greek mathematician, geographer, astronomer and poet, Eratosthenes (276-195 BC). There exists also the Sieve of Sundaram, an ingenious sieve for sorting out all the prime numbers up to a specified integer. It was discovered by the Indian mathematician S. P. Sundaram in 1934 [5].

The aim of this paper is to design a new sieving technique for generating primes from a list of natural numbers ending with any given odd digit with the exception of the digit 5. The method deletes composites from the list, leaving behind prime numbers ending with the given digit.

The remainder of this paper consists of two sections. The first section discusses how the method is used to obtain a sequence of primes ending with 3 from a sequence of natural numbers ending with 3. The second section deals with the use of the same approach in obtaining a sequence of primes ending with 1 from a sequence of natural numbers ending with 1.

2. Prime Sieve for Finding Primes Ending with 3

There is no conceivable pattern in the occurrence of the primes. After the number 2, primes can never be even and after the number 5, there are only four possibilities for the last digits of the primes–$1, 3, 7$ and 9. In this section we discuss how a list of primes terminating with 3 might be sorted out of a larger list of natural number terminating with the same digit 3.

It is known fact in number theory that the last digit of the product of two natural numbers ending with 1 and 3 or 7 and 9 is 3 for $1 \times 3 = 3$ and $7 \times 9 = 63$. This is illustrated in Tables 1 and 2.

Row	$\times$	03	13	23	33	$\cdots$
1	11	33	143	253	363	$\cdots$
2	21	63	273	483	693	$\cdots$
3	31	93	403	713	1023	$\cdots$
4	41	123	533	943	1353	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1. Multiplication Table for $N1 \times n3$

Row	$\times$	07	17	27	37	$\cdots$
1	09	63	153	243	333	$\cdots$
2	19	133	323	513	703	$\cdots$
3	29	203	493	783	1073	$\cdots$
4	39	273	663	1053	1443	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2. Multiplication Table for $N9 \times n7$

There are patterns in the tables that can help us spot out primes ending with 3. Let us commence with Table 1. In row 1, viz

$$33, \quad 143, \quad 253, \quad \ldots,$$

each number differs from the next by 110. In row 2, viz

$$63, \quad 273, \quad 483, \quad \dots,$$

each number differs from the next by 210. In row 3, namely

$$93, \quad 403, \quad 713, \quad \dots,$$

each number differs from the next by 310; and so on.

Let us now look at Table 2. In row 1, namely

$$63, \quad 153, \quad 243, \quad \dots,$$

each number differs by 90. In row 2, viz

$$133, \quad 323, \quad 513, \quad \dots,$$

each number differs from the next by 190. In row 3, namely

$$203, \quad 493, \quad 513, \quad \dots,$$

each number differs from the next by 290; and so forth.

With this pattern at hand, we can find all the primes ending with 3 up to any given natural number ending with 3.

Suppose we wish to generate all the prime numbers ending with 3 up to 393. First of all, we write down the list of every odd number ending with 3 up to 393. This is displayed in Table 3. We delete the composites of Table 3 by taking the

03	13	23	33	43	53	63	73	83	93
103	113	123	133	143	153	163	173	183	193
203	213	223	233	243	253	263	272	283	293
303	313	323	333	343	353	363	373	383	393

Table 3. Natural Numbers Ending with 3

following steps.

Step 1. Circle and cross out the first products

$$33, \quad 63, \quad 93, \quad \dots,$$

of rows $1, 2, 3, \cdots$ respectively of Table 1 (this sequence correspond to every 3rd number in the given list, by counting up from 33 in increments of 3). Table 4 displays this step.

3	13	23	~~33~~	43	53	~~63~~	73	83	~~93~~
103	113	~~123~~	133	143	~~153~~	163	173	~~183~~	193
203	~~213~~	223	233	~~243~~	253	263	~~272~~	283	293
~~303~~	313	323	~~333~~	343	353	~~363~~	373	383	~~393~~

Table 4. Step 1.

Step 2. Starting counting from the first circled number 33, cross out every 11th number in the list in increments of 11. The table becomes the one shown in Table 5 .

3	13	23	~~33~~	43	53	~~63~~	73	83	~~93~~
103	113	~~123~~	133	~~143~~	~~153~~	163	173	~~183~~	193
203	~~213~~	223	233	~~243~~	~~253~~	263	~~272~~	283	293
~~303~~	313	323	~~333~~	343	353	~~363~~	373	383	~~393~~

Table 5. Step 2.

Step 3. Starting counting from the second circled number 63, cross out every 21st number in the list in increments of 21. The table becomes the one shown in Table 6.

3	13	23	~~33~~	43	53	~~63~~	73	83	~~93~~
103	113	~~123~~	133	~~143~~	~~153~~	163	173	~~183~~	193
203	~~213~~	223	233	~~243~~	~~253~~	263	~~272~~	283	293
~~303~~	313	323	~~333~~	343	353	~~363~~	373	383	~~393~~

Table 6. Step 2.

We are through with Table 1.

We now proceed to Table 2. Delete the composites of this table by taking the following steps.

Step 1. Circle and cross out the first products

$$63, \quad 133, \quad 203, \quad \ldots,$$

of rows $1, 2, 3, \cdots$ respectively of Table 2 (this sequence correspond to every 7th number in the list, starting from 63). Table 7 displays this step.

3	13	23	~~33~~	43	53	~~63~~	73	83	~~93~~
103	113	~~123~~	133	~~143~~	~~153~~	163	173	~~183~~	193
203	~~213~~	223	233	~~243~~	~~253~~	263	~~273~~	283	293
~~303~~	313	323	~~333~~	343	353	~~363~~	373	383	~~393~~

Table 7. Step 1.

Step 2. Starting counting from the circled number 63, cross out every 9th number in the list in increments of 9. The table becomes the one shown in Table 8.

3	13	23	~~33~~	43	53	~~63~~	73	83	~~93~~
103	113	~~123~~	133	~~143~~	~~153~~	163	173	~~183~~	193
203	~~213~~	223	233	~~243~~	~~253~~	263	~~273~~	283	293
~~303~~	313	323	~~333~~	343	353	~~363~~	373	383	~~393~~

TABLE 8. Step 2.

Step 3. Starting counting from the circled number 133, cross out every 19th number in the list in increments of 19. The table becomes the one shown in Table 9.

3	13	23	~~33~~	43	53	~~63~~	73	83	~~93~~
103	113	~~123~~	~~133~~	~~143~~	~~153~~	163	173	~~183~~	193
~~203~~	~~213~~	223	233	~~243~~	~~253~~	263	~~273~~	283	293
~~303~~	313	~~323~~	~~333~~	~~343~~	353	~~363~~	373	383	~~393~~

TABLE 9. Step 3.

Step 4. Starting counting from the circled number 203, cross out every 29th number in the list in increments of 29. The table becomes the one shown in Table 10.

3	13	23	~~33~~	43	53	~~63~~	73	83	~~93~~
103	113	~~123~~	~~133~~	~~143~~	~~153~~	163	173	~~183~~	193
~~203~~	~~213~~	223	233	~~243~~	~~253~~	263	~~273~~	283	293
~~303~~	313	~~323~~	~~333~~	~~343~~	353	~~363~~	373	383	~~393~~

TABLE 10. Step 4.

We are through with Table 2.

The numbers not canceled in the list above are the primes ending with 3. These are displayed in Table 11.

3	13	23	43	53	73	83
103	113	163	173	193	223	233
263	283	293	313	353	373	383

TABLE 11. List of primes ending with 3.

3. Prime Sieve for Finding out Primes Ending with 1

The last digit of the product of two natural numbers ending with 1 and 1, 3 and 7, or 9 and 9 is 1 for $1 \times 1 = 1$, $3 \times 7 = 21$ and $9 \times 9 = 81$. This fact is made very clear in Tables 12, 13 and 14.

Row	$\times$	11	21	31	$\cdots$
1	11	121	231	341	$\cdots$
2	21	231	441	651	$\cdots$
3	31	341	651	961	$\cdots$
4	41	451	861	1271	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 12. Multiplication Table for $N1 \times n1$

Row	$\times$	07	17	27	$\cdots$
1	03	21	51	81	$\cdots$
2	13	91	221	351	$\cdots$
3	23	161	391	621	$\cdots$
4	33	231	561	891	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 13. Multiplication Table for $n3 \times N7$

Row	$\times$	09	19	29	$\cdots$
1	09	81	171	261	$\cdots$
2	19	171	361	551	$\cdots$
3	29	261	551	841	$\cdots$
4	39	351	741	1131	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 14. Multiplication Table for $N9 \times n9$

With the help of the patterns arising from these tables, we shall devise a technique for generating prime numbers ending with 1.

Suppose we wish to determine the primes ending with 1 and up to a given number, say 401 (See Table 15). Delete the composites of Table 15 by taking the following steps.

11	21	31	41	51	61	71	81	91	101
111	121	131	141	151	161	171	181	191	201
211	221	231	241	251	261	271	281	291	301
311	321	331	341	351	361	371	381	391	401

Table 15. Step 1.

Step 1. Circle and cross out the first products

$$121, \quad 231, \quad 341, \quad \ldots,$$

of rows $1, 2, 3, \cdots$ respectively of Table 12 (this sequence correspond to every 11th number in the list, starting from 121). Table 16 displays this step.

11	21	31	41	51	61	71	81	91	101
111	~~121~~	131	141	151	161	171	181	191	201
211	221	~~231~~	241	251	261	271	281	291	301
311	321	331	~~341~~	351	361	371	381	391	401

Table 16. Step 1.

Step 2. Starting counting from the first circled number 121, cross out every 11th number in the list in increments of 11. The table remains the same.

Step 3. Starting counting from the second circled number 231, cross out every 21st number in the list in increments of 21. The table remains the same and we are done with Table 12.

We now turn to Table 13. Delete the composites of Table 16 by taking the following steps.

Step 1. Circle and cross out the first products

$$21, \quad 91, \quad 161, \quad \ldots,$$

of rows $1, 2, 3, \cdots$ respectively of Table 13 (this sequence correspond to every 7th number in the list starting from 21). Table 17 displays this step.

11	~~21~~	31	41	51	61	71	81	~~91~~	101
111	~~121~~	131	141	151	~~161~~	171	181	191	201
211	221	~~231~~	241	251	261	271	281	291	~~301~~
311	321	331	~~341~~	351	361	~~371~~	381	391	401

Table 17. Step 1.

11	**~~21~~**	31	41	~~51~~	61	71	~~81~~	**~~91~~**	101
~~111~~	**~~121~~**	131	~~141~~	151	**~~161~~**	~~171~~	181	191	~~201~~
211	221	**~~231~~**	241	251	~~261~~	271	281	~~291~~	**~~301~~**
311	~~321~~	331	**~~341~~**	~~351~~	361	**~~371~~**	~~381~~	391	401

TABLE 18. Step 2.

Step 2. Starting counting from the circled number 21, cross out every 3rd number in the list in increments of 3. The table becomes the one shown in Table 18.

Step 3. Starting counting from the circled number 91, cross out every 13th number in the list 13. The table becomes the one shown in Table 19.

11	**~~21~~**	31	41	~~51~~	61	71	~~81~~	**~~91~~**	101
~~111~~	**~~121~~**	131	~~141~~	151	**~~161~~**	~~171~~	181	191	~~201~~
211	~~221~~	**~~231~~**	241	251	~~261~~	271	281	~~291~~	**~~301~~**
311	~~321~~	331	**~~341~~**	~~351~~	361	**~~371~~**	~~381~~	391	401

TABLE 19. Step 3.

Step 4. Starting counting from the circled number 161, cross out every 23rd number in the list in increments of 23. The table becomes the one shown in Table 20.

11	**~~21~~**	31	41	~~51~~	61	71	~~81~~	**~~91~~**	101
~~111~~	**~~121~~**	131	~~141~~	151	**~~161~~**	~~171~~	181	191	~~201~~
211	~~221~~	**~~231~~**	241	251	~~261~~	271	281	~~291~~	**~~301~~**
311	~~321~~	331	**~~341~~**	~~351~~	361	**~~371~~**	~~381~~	~~391~~	401

TABLE 20. Step 4.

Step 5. Starting counting from the circled number 231, cross out every 33rd number in the list. The table remains the same and we move to the third table.

Delete the composites of Table 20 by taking the following steps.

Step 1. Circle and cross out the first products

$$81, \quad 171, \quad 261, \quad \ldots,$$

of rows $1, 2, 3, \cdots$ respectively of Table 14 (this sequence correspond to every 9th number in the list starting from 81). Table 21 displays this step.

11	~~21~~	31	41	~~51~~	61	71	~~81~~	~~91~~	101
~~111~~	~~121~~	131	~~141~~	151	~~161~~	~~171~~	181	191	~~201~~
211	~~221~~	~~231~~	241	251	~~261~~	271	281	~~291~~	~~301~~
311	~~321~~	331	~~341~~	~~351~~	361	~~371~~	~~381~~	~~391~~	401

TABLE 21. Step 1.

Step 2. Starting counting from the circled number 81, cross out every 9th number in the list in increments of 9. The table remains the same.

Step 3. Starting counting from the circled number 171, cross out every 19th number in the list in increments of 19. Table 23 displays this step.

11	~~21~~	31	41	~~51~~	61	71	~~81~~	~~91~~	101
~~111~~	~~121~~	131	~~141~~	151	~~161~~	~~171~~	181	191	~~201~~
211	~~221~~	~~231~~	241	251	~~261~~	271	281	~~291~~	~~301~~
311	~~321~~	331	~~341~~	~~351~~	~~361~~	~~371~~	~~381~~	~~391~~	401

TABLE 22. Step 3.

Step 4. Starting counting from the circled number 261, cross out every 29th number in the list in increments of 29. The table remains the same and we come to the end of the sieving process.

The numbers not crossed out at this point in the list are all the prime numbers ending with 1 from 11 to 401. The list of primes ending with 1 from 1 to 401 is therefore as follows.

11	31	41	61	71	101
131	151	181	191	211	241
251	271	281	311	331	401

TABLE 23. Step 3.

CONCLUSION

This paper discussed a sieving technique for fishing out primes ending with 3 from a finite list of natural numbers ending with 3, and primes ending with 1 from a finite list of natural numbers ending with 1. The method can also be applied to spot out primes ending with 7 and 9 from the respective finite lists of natural numbers ending with 7 and 9.

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